

Title	On the Schur Indices of Algebraic Groups (SEMINAR ON PERMUTATION GROUPS AND RELATED TOPICS)
Author(s)	OHMORI, ZYOZYU
Citation	数理解析研究所講究録 (1978), 325: 135-139
Issue Date	1978-05
URL	<a href="http://hdl.handle.net/2433/104067">http://hdl.handle.net/2433/104067</a>
Right	
Type	Departmental Bulletin Paper
Textversion	publisher

## On the Schur indices of algebraic groups

Zyozyu OHMORI

Department of Mathematics,

Tokyo Metropolitan University

Let  $p$  be a fixed prime number and  $k$  be an algebraically closed field of characteristic  $p$ . Let  $G$  be a connected, reductive linear algebraic group defined over  $k$  and  $\sigma$  be a surjective endomorphism of  $G$  such that  $G_\sigma = \{x \in G \mid \sigma(x) = x\}$  is finite. I want to determine the rational Schur indices of  $G_\sigma$ . We assume that the centre  $Z$  of  $G$  is connected and that  $p$  is not a bad prime for  $G$ .

### 1. Semisimple and regular characters.

Let  $P$  be a Sylow  $p$ -subgroup of  $G_\sigma$ . The Gelfand-Graev character  $\Gamma_G$  of  $G_\sigma$  is the character of  $G_\sigma$  which is induced from any

linear character of  $P$  in "general position".  $\Gamma_G$  does not depend on the choice of such linear characters of  $P$ . That  $\Gamma_G$  is multiplicity-free is well-known. Each irreducible component of  $\Gamma_G$  will be called a regular character of  $G_0$ . Any irreducible character of  $G_0$  whose degree is coprime to  $p$  will be called semisimple. We note that if  $G$  is defined over a finite field  $\mathbb{F}_q$  ( $q = p^f$ ,  $f \geq 1$ ) and  $\sigma$  is the corresponding Frobenius endomorphism of  $G$ , the number of the regular characters of  $G_\sigma = G(\mathbb{F}_q)$  coincides with that of the rational ones and it equals  $|Z_\sigma| \cdot q^l$ , where  $l$  is the rank of  $G/Z$ .

Theorem 1 (R. Gow, Z. Ohmori) The character of  $G_\sigma$  which is induced from a linear character of  $P$  is rational. In particular,  $\Gamma_G$  is a rational character.

Corollary 2. The Schur indices of the regular or semisimple characters of  $G_0$  are all equal to one.

This follows from the next lemma.

Lemma 3. Let  $H$  be a finite group and  $\xi$  be a rational character of  $H$ . Then for any irreducible complex character  $\chi$  of  $H$ , the Schur index  $m_{\mathbb{Q}}(\chi)$  of  $\chi$  divides the intertwining number  $\langle \chi, \xi \rangle_H$ .

For example, any non-linear irreducible character of  $G_{\sigma} = GL(2, q)$  ( $G = GL_2$ ,  $\sigma$  is the Frobenius) is regular and the Schur indices of  $G_{\sigma}$  are one.

## 2. Application to $GL(n, q)$ and $U(n, q^2)$

Let  $G$  be a general linear group  $GL_n$  or a general unitary group  $U_n$ , both defined over  $\mathbb{F}_q$  and let  $\sigma$  be the corresponding Frobenius endomorphism of  $G$ . Then  $G_{\sigma} = G(\mathbb{F}_q)$  is isomorphic to the finite general linear group  $GL(n, q)$  or the finite general unitary group  $U(n, q^2)$ , respectively. We know

Theorem 4 (R. Gow) Any Schur index of  $G_{\sigma}$  divides 2.

Let  $u$  be a unipotent element of  $G_{\sigma}$ . There is a  $\sigma$ -stable subgroup  $L$  of  $G$  with the

following properties: (1)  $L$  is a connected, reductive group with connected centre; (2)  $p$  is a good prime for  $L$ ; (3)  $u$  is a regular unipotent element of  $L$ . Then by theorem 1 and lemma 3, we have

Theorem 5. Let  $\chi$  be an irreducible character of  $G_0$  and  $u$  be a unipotent element of  $G_0$ . Then  $\chi(u)$  is a rational integer and  $m_a(\chi)$  divides  $\chi(u)$ .

The value  $\chi(u)$  can be calculated by Green's work on the character table of  $GL(n, q)$  or Ennola's conjecture on that of  $U(n, q^2)$ , which has been proved for  $p > n$  and  $q \gg 0$  by Hotta, Kazhdan, Lusztig, Springer and Srinivasan.

Theorem 6 If  $G_0 = U(n, q^2)$ , assume that  $p > n$  and  $q \gg 0$  (if  $n \leq 5$ , this assumption can be dropped, and if  $G_0 = GL(n, q)$ , no assumption is needed). Then for any irreducible character  $\chi$  of  $G_0$ , there is a unipotent element  $u$  of  $G_0$  such that  $\chi(u)$  is equal to the  $p$ -part of the degree of  $\chi$  (which is a polynomial in

$q$  with rational integral coefficients) up to the sign.

Corollary 7 (Z. Ohmori) Let the assumption be as in theorem 6. Further assume that  $p > 2$ . Then all the Schur indices of  $G_0$  are one.

Remark that since any prime is good of  $G$ , any irreducible character of  $G_0$  of degree coprime to  $p$  has Schur index one (Cor. 2). Also note that Gow has proved that all the Schur indices of  $GL(2, q)$ ,  $GL(3, q)$  or  $GL(4, q)$  are equal to one.

### References

- [1] R. Gow, Schur indices of some groups of Lie type, J. Alg. 42 (1976), 102-120.
- [2] Z. Ohmori, On the Schur indices of  $GL(n, q)$  and  $SL(2n+1, q)$ , J. Math. Soc. Japan 29 (1977), 693-707.
- [3] —, On the Schur indices of reductive groups, Quart. J. Math. (Oxford) (2) 28 (1977), 357-361.
- [4] —, On the Schur indices of the finite unitary groups, to appear in Osaka J. Math.